THE CIRCLE

AND

STRAIGHT LINE.

PART THIRD.

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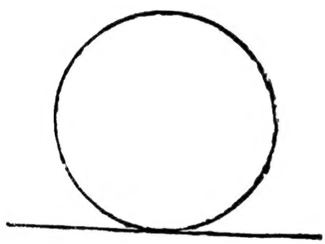

BY

JOHN HARRIS.

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THE CIRCLE AND STRAIGHT LINE.

'Prove *all* things; hold fast *that* which is *good*.'
St. Paul.

PART THIRD.

THEORY OF CURVATURE.

The Ultimate-Sine, and the arc of increasing curvature.—If a number of intermediate arcs be described with a proportionally increased radius, between *M.* and *n.*, between *n.* and *o.*, between *o.* and *p.*, &c., &c., respectively, it is evident that a continuous curved line drawn through the terminal points of all those arcs, will form a compounded arc of a peculiar character and which from its forming a connection as it were between the arc-length of the half-quadrant and the ultimate sine—i.e., the sine of the ultimate fraction of the half-quadrant—possesses much interest. We will here briefly indicate one mode in which the change in longitudinal magnitude in consequence of the elimination of curvature, as the fractional arc is diminished by successive bisections, may be investigated.

Taking the radius equal to 10. The length of the half-quadrant equals the sine-length increased by addition of 'the sine length divided by the units of radius less one.'

$$\frac{C}{S} = S + \frac{S}{R-1}$$

We thus obtain the magnitudinal length of the arc in units of radius. Now if this arc (half-quadrant) be bisected and the longitudinal magnitude of the remaining arc be duplicated, the half-quadrant is not reproduced

but an arc is obtained into which one half the curvature of the half-quadrant does not enter and in which the other half of the curvature belonging to the half-quadrant is divided over twice the quantity of linear extension.* The second arc therefore, although it is a magnitudinal duplication of the one-half of the first arc, is a lesser longitudinal magnitude than the first arc, because the magnitudinal duplication which has restored the direct linear extension has not replaced the curvature.

The process may be thus conducted:—

$\frac{C}{8}$.—The arc-length equals the sine-length increased by addition of the sine divided by the units in the radius less one.

$\frac{C}{16}$.—The arc-length equals the sine-length increased by addition of the sine divided by the units in the (duplicated) radius less one, multiplied by two.

$\frac{C}{32}$.—The arc-length equals the sine-length increased by addition of the sine divided by the units in the (duplicated) radius less one, multiplied by two, multiplied by four.

$\frac{C}{64}$.—The arc length equals the sine-length increased by addition of the sine divided by the units in the (duplicated) radius less one, multiplied by two, multiplied by four, multiplied by four. And so on, proceeding in like manner.

The successive divisors will be therefore.	1st	arc	$(R-1)$
	2	do	$(2R-1) \times 2$
	3	do	$(2R-1) \times 2 \times 4$
	4	do	$(2R-1) \times 2 \times 4 \times 4$
	5	do	$(2R-1) \times 2 \times 4 \times 4 \times 4$
	6	do	$(2R-1) \times 2 \times 4 \times 4 \times 4 \times 4$

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*If the half-quadrant, instead of bisected, were to be partially straightened into the same form as the half-arc, it would exceed in length the dupulicated half-arc by the difference of the curvature. (i.e., the difference between the curvature in the half-quadrant and in the arc of $22\frac{1}{2}$ degrees.)

Taking the successive sine lengths from the table given at page 54, Part Second, we obtain :—

E	*A*		*B*		*C*	*D*	
9	7·07107	+	·785674	=	7·856744	1 *R* –	1
38	7·65366	+	·201413	=	7·855080	4 *R* –	2
152	7·80361	+	·051339	=	7·854951	16 *R* –	8
608	7·84137	+	·012897	=	7·854597	64 *R* –	32
2432	7·85082	+	·0032281	=	7·854055	256 *R* –	128
9728	7·85319	+	·0008073	=	7·853997	1024 *R* –	512
38912	7·85379	+	·0002018	=	7·853991	4096 *R* –	2048
155648	7·85393	+	·0000504	=	7·853984	16384 *R* –	8192
622592	7·85396	+	·0000126	=	7·853978	65536 *R* –	32768

In this table, the divisors in the first column *E*. result from the last column *D*. Under *A*. are the sine lengths. Under *B*. are the quotients of the division of the sine lengths by the numbers under *E*, which quotients added to the sine lengths give the arc-lengths in column *C*.

It becomes evident that, by continuing the table, the quantities in column *B*., which represent the curvature, would become continually less, and the quantities under *A*. and *C*. would approximate more closely. In reference to column *B*. it may be observed that the third quotient is rather greater than the fourth part of the 2nd, that the 4th is a little more than the fourth part of the 3rd, the 5th a very little more than the fourth part of the 4th and that, of the remaining quotients, each is the fourth part of that preceding it, so far as the figures are carried; a greater number of decimals would show each of the successive lesser quantities to slightly exceed the one-fourth of the quantity preceding it, but the approximation to the one-fourth becoming continually closer as the process is carried further. To appreciate the meaning of this it is necessary to remember that the chord of each arc, which chord equals the sine of the next succeeding arc, is the square root of the square of the versed sine added to the square of the sine; and that the versed sine be-

longing to each arc is always greater than the one-half of that belonging to the arc preceding it; but as the terminal radius approaches the perpendicular (i.e., as the curvature becomes more nearly eliminated), the versed sine of each arc approximates more and more closely to the one-half of that belonging to the arc preceding it.

The process thus suggested is put forward as worthy of attentive consideration, and for the purpose of illustrating the subject; but we wish it to be particularly observed that (for the present) it stands by itself, nothing else is based upon it; it is not put forward as a demonstration, nor can we (at present) vouch for the correctness of the process or for the accuracy of the figures.

THE ARC OF INCREASING CURVATURE—This arc is distinguished from the arc of the arc-length (that is from an arc described with the length of the arc as a radius), and illustrated in Figs. 21 and 22. In Fig. 21, *X X* represents the arc described with the arc-length for a radius, and *S. n. o. p.* the arc of increasing curvature. In Fig. 22, the relation of the same two arcs to the construction of the circle is illustrated on a larger scale.

(*Note.—With these Figures compare Figs.* 20 *and* 25.)

THE TANGENTIAL LINE *B. D.*—

(Fig. 23 is a development of Fig. 12.)

Fig. 23. Bisect the radius *A.B.* in *a.*, and with centre *a.*, and radius *a.B.* describe the quadrant *B.K.*; bisect the quadrant in *m.* and bisect the half-quadrant *B.m.* in *'n.*; through *'n.* draw *a.n.* intercepting the line *B.D.* at *n.* Bisect also *C.D.* in *b.*, and with radius *b.D.* describe the quadrant *D.K.*; bisect *D.K.* in *M.* and bisect the half quadrant *D.M.* in *'N.*; through *'N.* draw *b.N.* intercepting the line *B.D.* at *N.* And bisect the arc *B.S.* in *'p.*, and through *'p.* draw *A.p.* With centre *C.* and radius *C.D.* describe *D.A.* bisected by the line *R.T.* in *Z.*, bisect *D.Z.* in *'P.* and through *'P.* draw *C.P.* intercepting *B.D.* at *P.*

We now have a division of the definite line *B.D.* into certain known quantitive magnitudes. Because *B.n.* is the tangent of the arc *B.'n.*, and *B.p.* is the tangent of the similar greater arc *B.'p.* which is tw ce the magnitude of *B.n.*, *B.p.* is twice the length of *B.n.*, and *n.p.* is equal to *B.n.* At the opposite extremity the similar points *D.N.* and *D.P.* are similarly related, and *N.P.* is equal to *D.N.*, which is equal to *B.n.*

Also because *B.q* is (equivalent to) the sine of *B.m.*, *B.q.* is the half of *B.W.* which is (equivalent to) the sine of the similar greater arc *B.S.*

Taking the radius *A.B.* = 10, the numerical values will be :—

$$B.K = 5{\cdot}00000 \quad = \quad D.K.$$
$$B.n. = 2{\cdot}07107 \quad = \quad D.N. = n.p.$$
$$B.p. = 4{\cdot}14214 \quad = \quad D.P.$$
$$B.q. = 7{\cdot}07107 \div 2 \quad = \quad 3{\cdot}535535 = B.Q.$$
$$P.K. = 5 - 4{\cdot}142136 \quad = \quad {\cdot}857864 = p.K.$$
$$B.N. = 10 - 2{\cdot}071068 \quad = \quad 7{\cdot}92893 = 3Bn + 2\,p.K.$$

Having in mind the relative magnitudes of the primary lines determined by trigonometry, these values present themselves at once as evident; we may then obtain, from and by means of these, the quantitive values of other lines, and may also, thus quantitively, determine the geometrical (magnitudinal) relation of other lines; that is to say—the inductive reasoning, by means of which further knowledge of the relation of the parts of the circle to each other is to be obtained, may be based upon the facts belonging to the science of 'Number and Quantity' in place of those belonging to 'Form and Magnitude.

For example :—

Because $B.n = 2{\cdot}07107$, $D.K = 5$, and $B.W = 7{\cdot}07107$

Therefore $K.n. = W.D. = S.W. = S.T. = K.N. = 2{\cdot}92893$

And, $2{\cdot}92893 \times 7{\cdot}07107 = 20{\cdot}7107\ldots.$

Therefore, $S.W. \times B.W. = B.n. \times B.D.$ That is—the versed sine × the sine of the half-quadrant = the tangent to the arc of $22\frac{1}{2}°$ × the tangent to the half-quadrant.

Because $K.p. = B.K. - B.p. = (5 - 4{\cdot}142146) = {\cdot}857864$
therefore $K.p \times 10 = 10K.p = (K.n)^2 = 2{\cdot}92893^2 = 8{\cdot}57864$
$= (S.T)^2 = (S.W)^2 = (P.p. \times 5)$.
And $P.p. = 2\ K.p = 1{\cdot}71572$.
But the square of $4{\cdot}142136 = 17{\cdot}1572$
Therefore $P.p. \times 10 = (S.D)^2 = (B.p)^2 = (D.P)^2$, &c., &c.
$B.P = B.K + K.P = 5{\cdot}857864 = 2\,WD = 2Kn$.
Now the tangent to the half of the bisected half-quadrant duplicated, $= 8{\cdot}28428 = 2\ B.p = 2\ D.P.$ And $(2\ B.p - B.W) = (K.n - 2K.p.) = (8{\cdot}28428 - 7{\cdot}07107) = (2{\cdot}92893 - 1{\cdot}71572) = 1{\cdot}21321$, &c., &c.

A fact of a primary character, which has not, we believe, been as yet applied in the art of computation, or otherwise utilized, is exhibited in the following:—

THEOREM.—If the quadrant of a circle be bisected and the secant to the half-quadrant be drawn through the point of bisection; and if the radius be produced through the centre of the circle and the produced radius be made twice the length of the original radius—that is, equal to the diameter of the circle—and a line be drawn joining the extremity of the produced radius and the extremity of the tangent; then shall the sine of the arc cut off from the quadrant by the line so drawn have to the tangent of the half-quadrant the ratio of four to five.

Fig. 24. With centre A. and radius $A.B.$ describe the quadrant $B.C.$, and bisect $B.C.$ in the point S. Draw the tangent $B.D.$ and secant $A.D.$ Join $D.C.$ and $A.C.$ Produce the radius $A.B.$ through A. and make $E.B.$ the production of $A.B.$ double the length of $A.D.$ Join $E.D.$ intersecting the quadrant at K. and bisecting the line $A.C.$ at g. Bisect the line $C.D.$ in the point d. and join $B.d.$ intersecting the quadrant at the same point K. From K. draw $K.Q.$ perpendicular to $B.D.$ and intercepting $B.D.$ at Q. From Q. draw $Q.H.$ parallel to $A.D.$ and intercepting $A.B.$ at the point H. From H. draw $H.P.$ parallel to $A.C.$ and equal in length to $B.Q.$ Join $P.K.$

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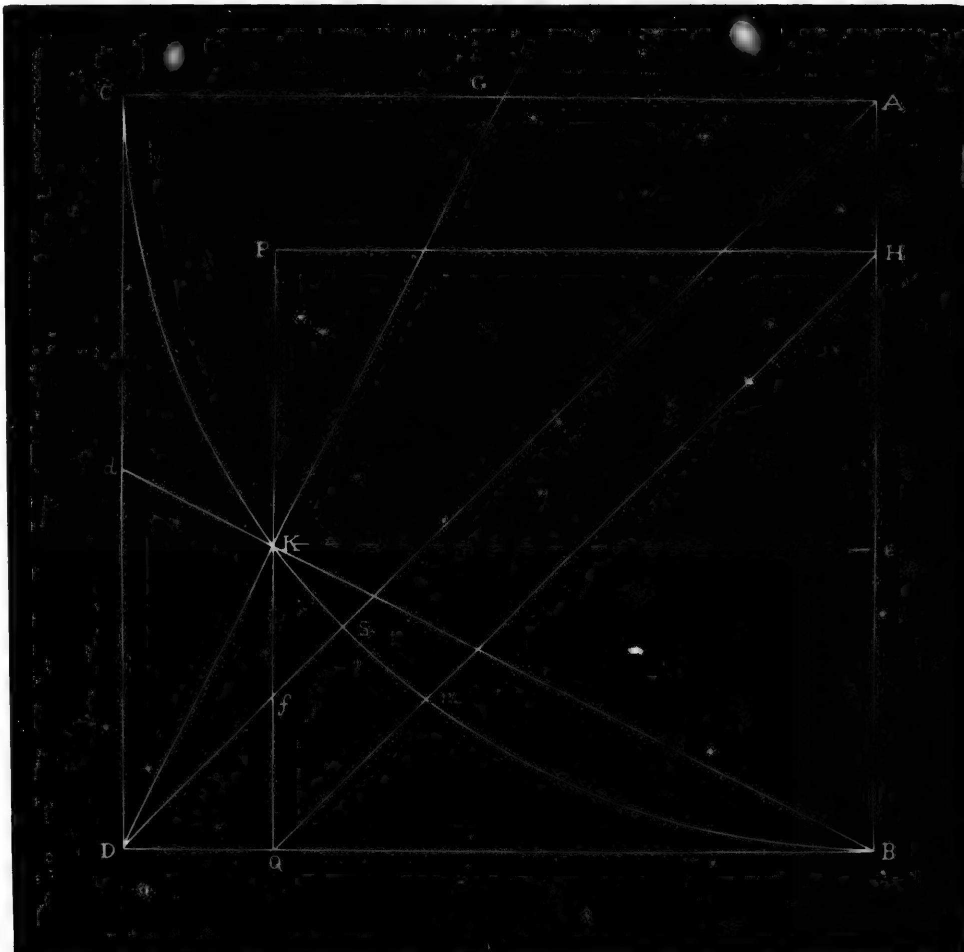
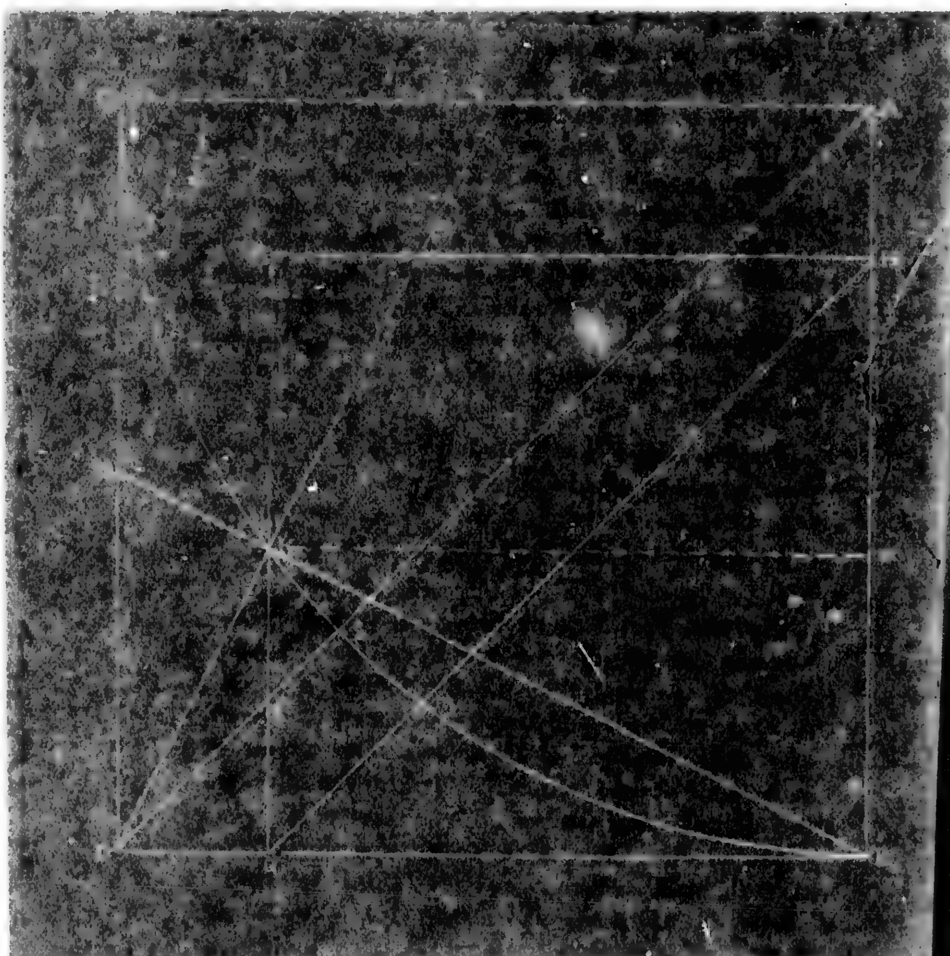
FIG. 24.



[illegible] parallel to A.C. the triangle [illegible] A.C.D. And since the [illegible] line B.d. which inter- [illegible] line P. [illegible] at the point [illegible] triangle H.K.Q. [illegible] is equal to twice [illegible] therefore the line [illegible] B.D. at [illegible] manifestly [illegible]

[illegible]

[illegible] at right [illegible] the production [illegible] parallel to Q.H. draw D.A., [illegible] B.H. at [illegible]. With centre A, [illegible]



Because *E.B.* is twice the length of *B.D.*, and *B.D.* twice the length of *d.D.* the triangle *B.d.D.* is similar to the triangle *E.D.B.* But the angles *B.D.d.* and *E.B.D.* are both of them right angles, therefore the line *B.d.* is at right angles to the line *E.D.* Now since *H.Q.* is parallel to *A.D.* and *H.P.* is parallel to *A.C.* the triangle *H.P.Q.*, is similar to the triangle *A.C.D.* And since the line *B.d.* bisects *C.D.*, the same line *B.d.* which intersects *P.Q.* at *K.* also bisects the line *P.Q.* in the point *K.* [And the line *B.K.* (part of *B.d.*) has been shown to be at right angles to the line *K.D.* (part of *E.D.*) therefore the triangle *K.D.Q.* is similar to the triangle *B.K.Q.* and *K.Q.* is the half of *P.Q.*] And *P.Q.* is (by the construction,) equal to *B.Q.*, therefore *B.Q.* is equal to twice *K.Q.*, and *K.Q.* is equal to twice *D.Q.*; therefore the line *B.Q.* has the same ratio to the whole line *B.D.* as 4 : 5. Draw *K.e.* the sine of the arc *B.K.* *K.e.* is manifestly equal to *B.Q.*

Problem.—Upon a given straight line it is required to describe an arc containing 45 degrees, such that if the tangent of the arc be divided into five equal parts, four of those parts shall be together equal to the given straight line.

Fig. 24.—Let *B.Q.* be the given straight line, it is required to describe the arc upon *B.Q.* From the point *Q.*, perpendicular to *B.Q.*, draw *Q.P.*, equal in length to *B.Q.*; and from *B.*, perpendicular to *B.Q.*, draw *B.H.*, also equal to *B.Q.* Join *H.Q.* Bisect *P.Q.*, in the point *K.* and join *B.K.* Produce *B.Q.* through *Q.* and produce *B.H.* through *H.* (Through *K.*, at right angles to *B.K*, draw *K.D.* intercepting the production of *B.Q.*, at *D.* From *D.* parallel to *Q.H.** draw *D.A.*, intercepting the production of *B.H.* at *A.* With centre *A.*,

* Or, join *B.P.*, and from *D.* at right angles to *B.P.* draw *D.A.*, intercepting the production of *B.H.* at *A.*

and radius *A.B.*, describe the quadrant *B.S.C.*, intersecting *D.A.* in the point *S.* *B.S.* shall be the required arc.

The demonstration to this solution is substantially contained in the demonstration to the theorem.)

Bisect *K.Q.* and, through *f.* the point of bisection, at right angles to *B.P.*, draw *D.A.* intercepting the production of *B.Q.* at *D.*, and intercepting the production of *B.H.* at *A.* With centre *A*, and radius *A.B.* describe the quadrant *B.S.C.* bisected by the line *D.A.* in the point *S.* *B.S.* shall be the required arc.

Because *D.Q.f.* is a right angle and *f.* bisects *K.Q.* *Q.D.* is equal to the one half of *K.Q.* But *K.* bisects *P.Q.*. and *P. Q.* equals *B.Q.*, therefore *Q.D.* is equal to the one fourth part of *B.Q.* Wherefore if *B.D.*, the tangent of the arc *B.S.*, be divided into five equal parts, four of those equal parts, shall together contain the given straight line. Q.E.D.

We have now to make known a discovery which will facilitate the investigation of the structural characteristics of the circle and which may become hereafter of much utility and value in simplifying trigonometrical (cyclometrical) processes. We propose for the present to treat it as a fundamental and, in a measure, independent fact, belonging to the plan of the circle. The discovery is stated in the following:—

Theorem.—That if the radius of the half-quadrant be produced through the centre of the circle until it become equal to the versed sine (of the half-quadrant) multiplied by ten, and with this radius an arc be described terminated by a line drawn from the centre of the greater circle—*i.e.*, from the point at the (central) extremity of the radius—through the point at the extremity of the half-quadrant, the tangent to the arc last described (of greater magnitude), cut off by the line so drawn from the tangent of the half-quadrant, will be equal to the arc-length of the half-quadrant.

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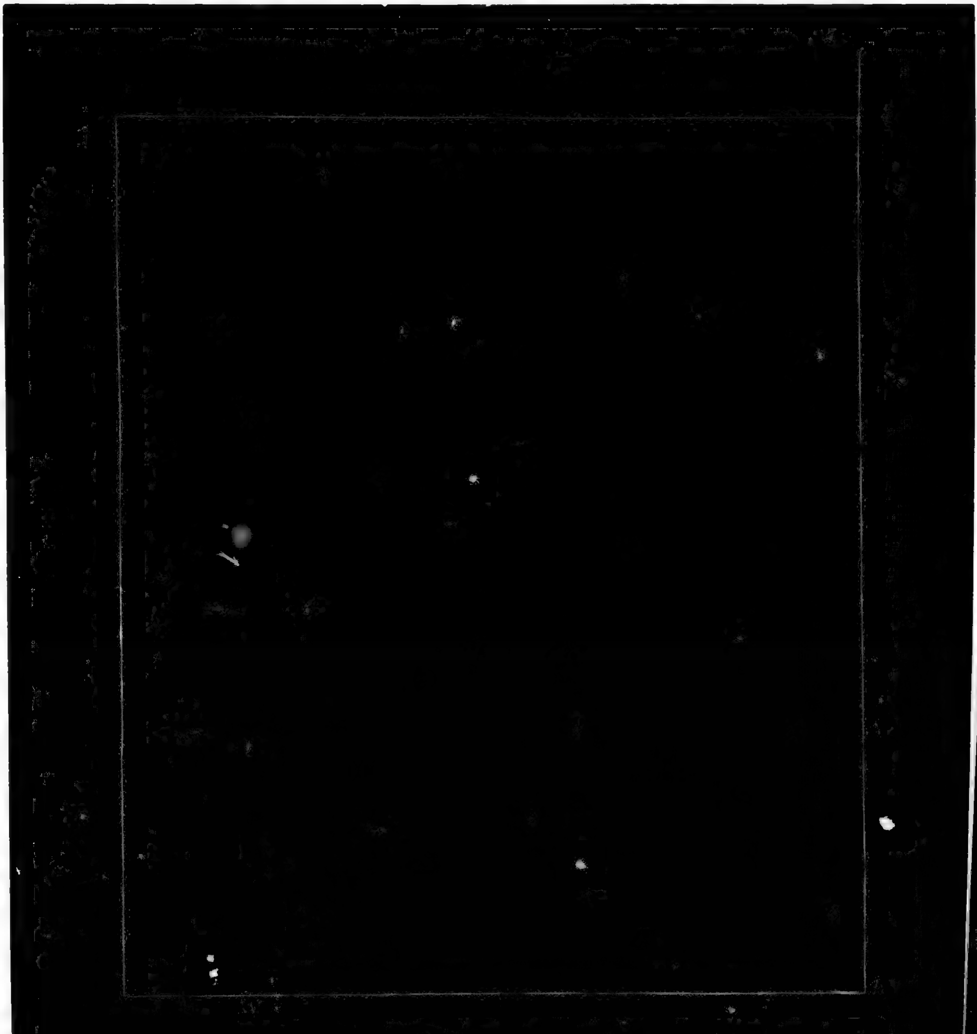
FIG. 25.



[illegible] $B.D.$ [illegible] and make [illegible] $S.W.$ [illegible] R, through S [illegible] $B.D.$ in the point L. $B.T.$ [illegible] are $B.S.$*

[illegible] (quantitive). The [illegible] length of $B.S.$, the length of the [illegible] [illegible] used [illegible]

[illegible] $B.S.$

[illegible] $A.F.$ [illegible]

[illegible] $S.W.$ [illegible]

[illegible] dius, $B.E.$ [illegible]

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[illegible] $B.S.$

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[illegible] arc of [illegible] $B.S.$ [illegible] half

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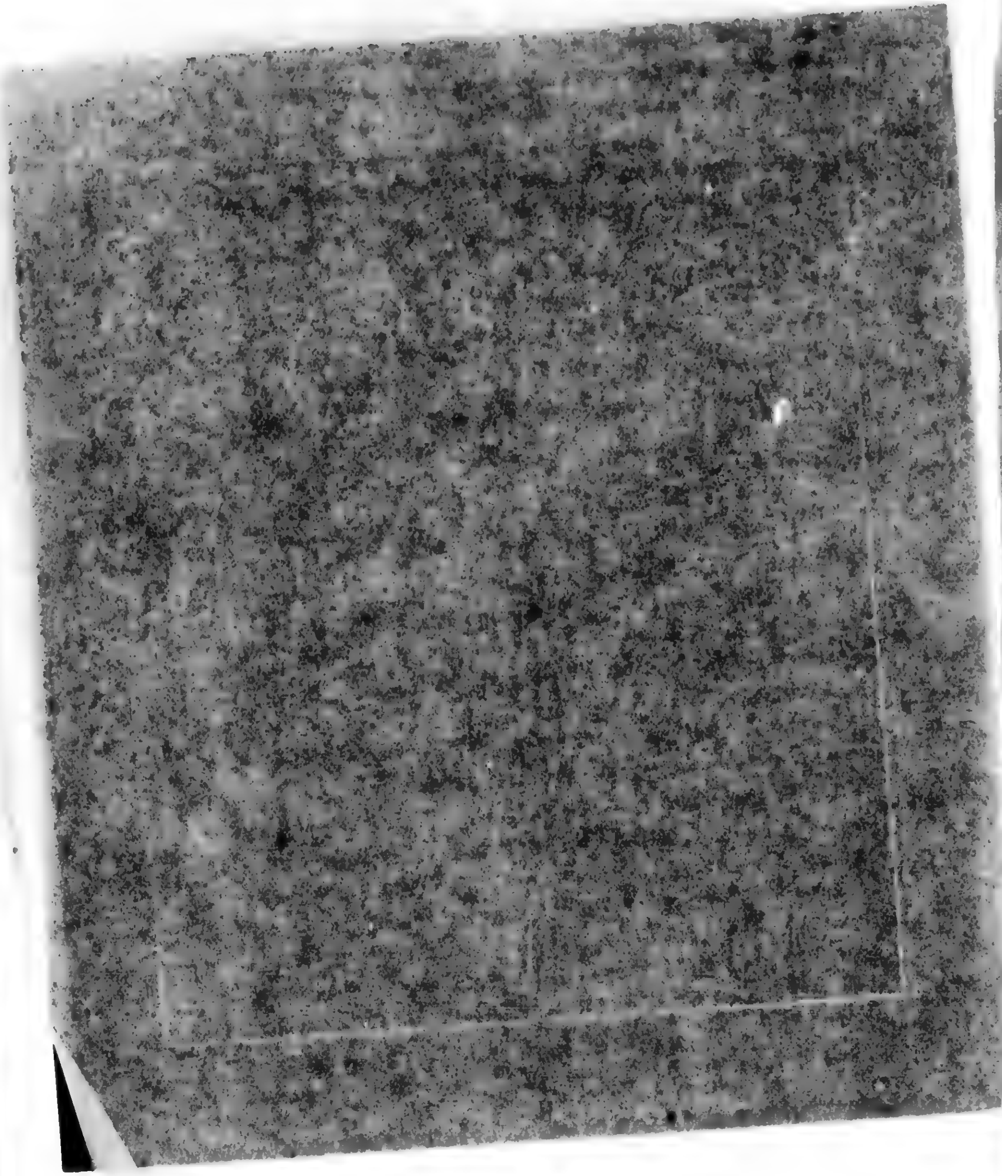


Fig. 25.—Let *B.S.* be the half-quadrant of which *A.B.* is the radius; produce *A.B.* through *A.* and make *R.B.* equal to ten times *S.W.* From *R.* through *S.* draw *R.X.* intersecting *B.D.* in the point *X.* *B.X.* shall be equal to the arc *B.S.**

Demonstration. (Quantitive). The length of the sine, *S.N.* the arc-length of *B.S.*, the length of the radius *R.B.* and of the versed sine *S.W*, are known, we have to obtain the length of *B.X.*

The radius, $A.B. = 10.$

" *v.* sine, $S.W. = 2{\cdot}92893.$

" radius, $R.B. = 10\ S.W = 29{\cdot}2893.$

$R.N. = R.B - S.W = (29{\cdot}2893 - 2{\cdot}92893) = 26{\cdot}3603.$

Now the lesser triangle *S.W.X.* is similar to the greater triangle *R.N.S.*

Therefore $W.X. : N.S. :: S.W. : R.N.$

$W.X. : 7{\cdot}07107 :: 2{\cdot}92893 : 26{\cdot}3603.$

$W.X. = {\cdot}78567$ But $N.S. + W.X. = B.X.$

Therefore, $7{\cdot}07107 + {\cdot}78567 = 7{\cdot}85674.$

And *B.S.* the arc-length also equals $7{\cdot}85674$. Wherefore the tangent *B.X.*, of the arc of greater magnitude is equal to the arc length *B.S.*, of the half quadrant. (Q.E.D.)

Corollary. Hence manifestly *W.X.* = the difference of the sine and arc-length of the half-quadrant.

The Ultimate tangent—We have objected to Legendre's circumscribed polygon—which eventually almost coincides with the sine—that it cannot be a simple continuous figure but must be fragmentary and compound; and we based this conclusion primarily and directly on the manifest necessity that a circumscribed polygon if continuous must be greater than the circle which it surrounds. We propose now to show that Legendre's circumscribed polygon is in fact compounded of the minute ultimate tangents

* Or:—From *X*, through *S*, draw *X.Y.R.*, intercepting the production of *B.A.*, at *R.* *R.B.* shall be ten times the length of *S.W.*

separate from each other but arranged together in the form of a polygon. To show this we will consider the question whether the tangent to an arc, if that arc be an indefinitely small fraction of the circle, is necessarily greater than the arc itself. It is at present held that the tangent is always greater than the arc by the evidence of trigonometry, thus:—Let *A.B.* be the arc, and *A.d.*

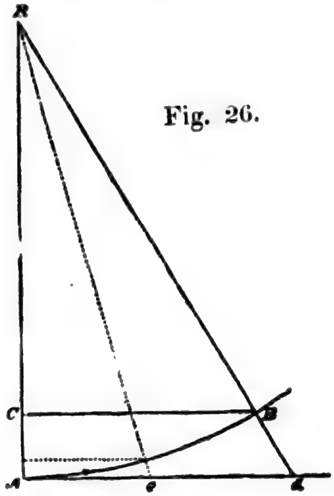


Fig. 26.

the tangent. The arc is bisected; and from the point *e*, *e.B.* is drawn perpendicular to *R.B.*, intercepting *R.B.* at *B.* Now the triangle *d.e.B.* is similar to the triangle *d.R.A.*, and the side *d.e.* is greater than the side *B.e.* of the lesser triangle, in the same ratio that the side *d.R.* is greater than the side *A.R.* of the greater triangle. Since evidently so long as it is possible to assign any quantity of magnitude to the remaining arc, it is possible to imagine the same process to be performed, it is supposed to follow that, however small a fraction of the circle an arc may be, so long as there be an arc, the same reasoning demonstrates the least tangent belonging thereto to be greater than the arc, because it is thereby shown that the outer tangent (as *A.d.*) is greater than the two

interior tangents (as *A.e.* and *e.B.*) taken together; and it is assumed as manifest that the two interior tangents taken together are always greater than the arc.

We submit that such conclusion is shown to be erroneous by the following: Let *A.B.* be a given straight line. At each extremity of the line draw a perpendicular *A.R.* and *S.t.* (*t.* being the same point as *B.*) Let

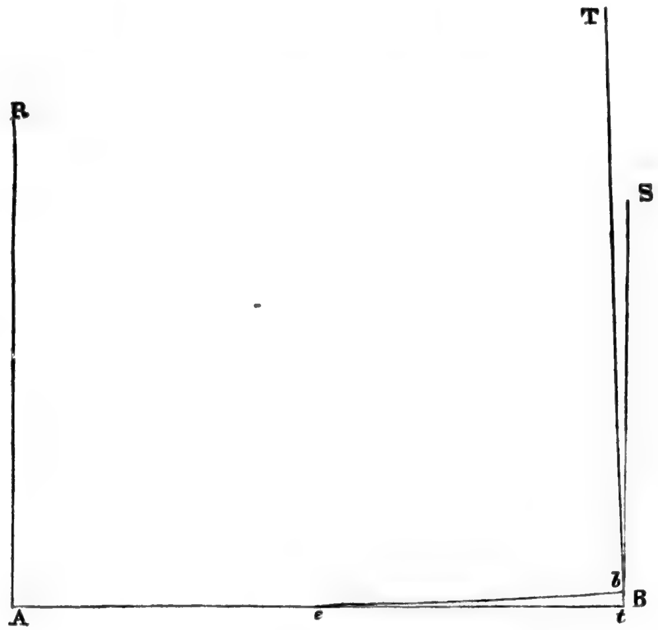


Fig. 27.

the same line *A.B.* be supposed to be curved* into the form of an arc containing a very small fraction of the circle.

* If the expression 'bending' or 'curving the line' be objected to, then, let a fractional arc equal in length to the given straight line, and belonging to a circle of any very great magnitude, be applied upon the straight line *A.B.* in such wise that the point at one extremity of the straight line shall coincide with the point at one extremity of the arc, &c., &c.

It is manifest that the line being curved will be less in direct length, measuring horizontally between the perpendiculars, than when it was straight, and, although the same line and of the same actual length as when straight, it will not now extend quite so far from *A.* as to reach the line *S.t.* Let any point *b.*, less distant than *S.t.* from *A.*, by a very small quantity of space, be the terminal extremity of the arc. It is evident that, however small the distance of the point *b.* from the line *S.t.*, if a line be drawn from the point *t.* through *b.* and the line so drawn be produced until it meets the production of *A.R.* through *R.* then, if the line so drawn may be correctly considered the secant of the arc, the tangent is *A.t.* and is not greater than the arc. The question, therefore, assumes the form :—whether the line *T.t.* drawn through the point *b.* at the extremity of the arc, is at right angles to the extremity of the arc. If it be not at right angles, the tangent may be greater than the arc; because the point *t.* at the extremity of the secant might then fall beyond the distance *B.* from *A.* On this question, however, we have the evidence of trigonometry as given in the polygon computation of Legendre (already quoted) and others of the like kind, and which evidence is decisive because it is shown thereby that ultimately the tangent-length almost coincides with the sine-length, and, since the sine-length is manifestly less than the arc-length, it is evident that in fact the extremity of the secant to the ultimate arc does not fall beyond the arc-length on the tangential line, and, since the ultimate tangent-length cannot be less than the arc-length, it appears safe to conclude that the tangent-length actually coincides with the arc-length of the ultimate arc.

Let *D.E.A.F.B.* (Fig. *a*) be four equal divisional parts of the half-quadrant *D.B.* Fig. *b.* represents one of those equal divisional parts on a larger scale, the straight line being equal to *A.e.* Now if we apply four straight lines each equal to *A.e.* as tangents to the divisions of

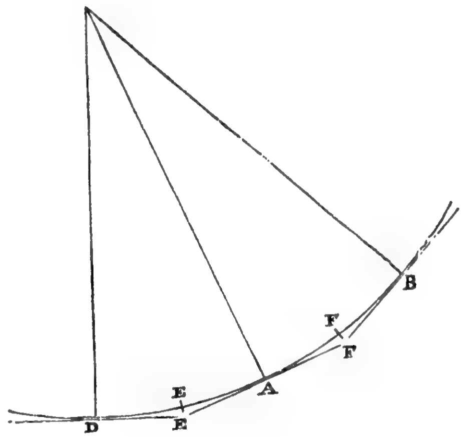


Fig. a.

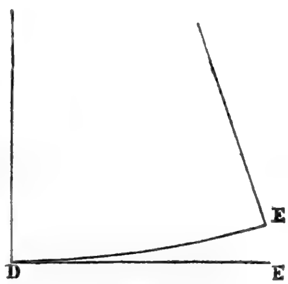


Fig. b.

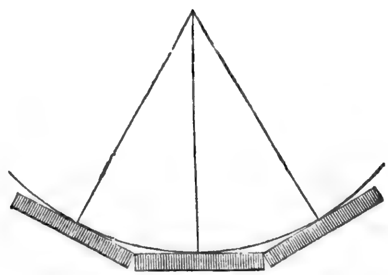


Fig. c.

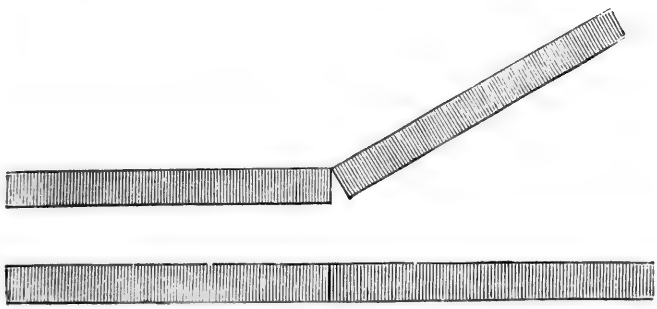


Fig. d.

the half-quadrant, as shown in Fig. *a*. it is evident that they cannot meet each other at the extremities which are not in contact with the circle. If they had sufficient length to meet each other, angles would be formed at the points *E*. and *F*., but, the length being insufficient, spaces are left at those points. Supposing the four lines to be again divided by bisection, and the eight half-lines to be arranged as tangents to the circle, there will be four spaces instead of two where the extremities of the lines will not quite meet each other, but they will be much more nearly in contact than before. If the lines be in like manner repeatedly divided, there will be eventually obtained a polygon, formed by the ultimate tangents which will then be in contact, the extremities of each with the extremities of those next thereto. What are the conditions of that contact ? If we accepted the dogma of Euclid, we should have to consider the contact as complete and the polygon as a simple figure, and to accept the sum of the ultimate tangent-lengths as the measurement of the circle; but we reject the dogma and insist on a real circle containing area and, consequently, containing breadth.

We have, accordingly, to enquire how the lines are divided. Is the section oblique or perpendicular to the length of the line ? Now this is readily answered—because the ultimate tangent-lines, derived from continued bisection, are divisions of a straight line evenly divided at right angles to its length. Therefore, by placing the divisional lines of Fig. *a*. in the positions, relative to each other, which they would occupy as tangents to the circle, and considering the rectangular extremities of the one with reference to the rectangular extremities of the next the actual conditions of the contact between them will become manifest. Fig. *c*.

The fragmentary tangent-lines compounding the circumscribed) polygon are, therefore, in contact with respect to the inner surface only; but, if a straight line be

compounded of the fragmentary lines, the contact will be complete—that is, perfect with respect to both surfaces—and the original straight line will be restored.

This case may be taken by itself. The fact of the ultimate arc equal in length to its tangent, and of the half-quadrant greater than the sum of all its ultimate tangents, presents itself for acceptance. That it is indisputable and must be accepted seems to us evident. But, what then: if there be some persons to whom this fact appears irreconcilable with the facts (as now taught) of trigonometry?

In concluding this treatise it may be not out of place to consider a little the meaning of the familiar statement (phrase) that one fact cannot contradict another. Probably no one at the present time with any pretension to scientific knowledge or education will directly deny or express any doubt as to the truth and certainty of such statement in its simple form; but the question is as to absolute certainty in the strictest and most comprehensive sense; and in the writings of men of scientific reputation, at the present time, opinions are expressed, theories propounded, and statements made, which, if referred inferentially to their basis, make apparent that, in the minds of the writers, no clear and reasonable conviction has been arrived at that, as a scientific certainty, one fact may not or cannot contradict another; i.e., be otherwise than in harmony with other facts. We call attention here to this circumstance because we believe that, in many cases, if the writer distinctly understood the nature of the conclusions involved or of the reasonable inferences belonging as corollaries to those theoretical opinions and statements, they would be reconsidered and, being found to be unreasonable, would not be put forth. This circumstance, which must needs be regarded as of the gravest importance, may in part at least, be a consequence of the increasing neglect

of mathematical reasoning—by which we mean, strictly lawful and (therefore) correct reasoning—as a necessary element of educational training. A consequence of the want of such training is a non-recognition by the mind of the necessity of a primary basis, either securely established or undoubtingly believed in, upon which the theory or statement must be actually or presumably based. Practically the effect displays itself in the opinion now more or less openly expressed by scientific teachers that Theology is not an absolutely necessary and fundamental part of Science, but that, on the contrary, Science and Theology, if not actually antagonistic, should be kept quite distinct and separate.

This separation of Science from Theology appears to be concurred in, and even supported, by very many of those who teach that the fundamental basis of Science must be found in Theology. They, nevertheless, accept the title of theologians, as distinguished from, and as in some degree antagonistic to that of, men of Science. Moreover the dislike of the man of Science to Theology, is perhaps, more than reciprocated by the theologian whose belief in the primary truth of Theology is oftentimes a belief fearful and apprehensive lest the progress of Science should discover and make known some adverse truth irreconcilable with the truth of Theology.

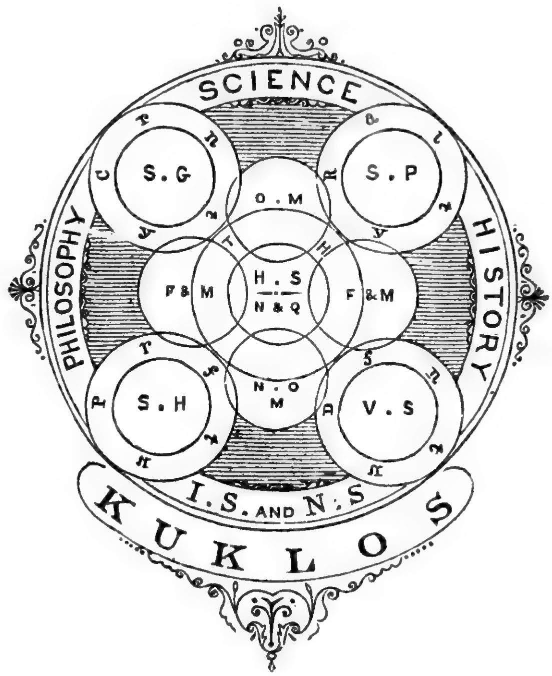
That this view of the case is adopted by all or by nearly all scientific men or theologians we are not supposing. The exceptions on both sides are probably in the aggregate very many, but the question is, taken all together, what is the conclusion or state of feeling now entertained and held by the educated public generally; and in what direction are the doctrines now taught guiding those who are to succeed us?

Assuming the existing state of things to be substantially as we have suggested, the apprehensive belief of the theologian which fears the facts and discoveries of science is not a belief in a scientific sense; such a fear

evidently includes a doubt as to whether one fact may not contradict and be irreconcilable with another or with some other facts. Nor can it help matters to say that the truth of theology is different from the truth of science, because, if it be meant thereby that the truth of the one can be otherwise than in perfect harmony with the truth of the other, the entertaining or admiting in any degree such a supposition is the making over Theology to Metaphysics which is expressly the antagonist of sound Theology as being the essential basis of all sound Science. How stands the case on the other side? If the question was put: what is the especial principle contended for and upheld by modern science? the answer most generally concurred in would be—'intellectual freedom,' the right of each one to exercise his intellect in his own way, without let or hindrance. What does that mean? It means *intellectual lawlessness;* intellectual anarchy; the right of each one in matters which concern others as well as himself in the most important and momentous sense, to do as seemeth good in his own eyes; the demand of men, who recognise the imperious necessity of laws and submission to those laws in matters of comparatively little importance, to be free from the restraint of all law in matters of the greatest importance. It means the claim of some men to be allowed to impede sound education, to pollute things sacred, to contaminate all true knowledge, to teach the most pernicious doctrines, and to deceive and confound those who look to them for instruction.

Against this claim of so-called intellectual freedom, whether it be hesitatingly expressed, merely as a dislike to mixing up theology with science, or boldly and defiantly asserted as a determination to be intellectually unrestrained by any laws or rules, we distinctly protest and earnestly caution those whom we may in any degree influence. It may be said—let public discussion and progressive education put right what is wrong, we

cannot expect to have science quite perfect at any time; that which is true is stronger than that which is false—increased knowledge and discussion may be relied upon to eliminate error. No: such reasoning is not itself based on fact; and such a course is not to be relied on to eliminate error; on the contrary if allowed to proceed, its effect would be to vitiate education and to deteriorate knowledge more and more, until, at length, the education would become an altogether evil education, and the knowledge an essentially unsound and destructive knowledge. Truth is stronger, much stronger, than falsehood, but a mixture of that which is true and that which is false is not truth. Neither is it fact that error can be eliminated and got rid of by discussion in which sound and unsound knowledge are mingled together, in which the reasoning is not strictly lawful and not properly based on that which is true and certain. Such discussion is the weapon with which Untruth, wearing the mask of Science, can best succeed in destroying that which is true, and confounding those who love the truth. A human science which does not distinctly recognize the primary truths of Theology as its ultimate basis, is not based on reality; it has not and cannot have any actual and secure foundation. If the Science of England is not so based, no matter what seeming progress may for a time be made, whenever the trial comes it will be as the house built on the shifting sand; and, if not destroyed by sudden catastrophe, will eventually become a ruin, together with the civilization which rests upon it.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APPENDIX.

Figs. 27, 28, 29, 30, are additional illustrations of the complementary arc of absolute curvature, of the arc of increasing curvature, and of the increasing sine-length.

It may be observed that in some of these figures the same letter of reference, $\mathfrak{X}.$, indicates the arc length and the ultimate sine-length. On the scale of these figures the two points fall so closely together as to appear almost identical, the difference being only about three parts in eight thousand, and an attempt to show them separately in the same figure would confuse the lines and letters.

It is supposed that, with the explanations which have preceded them, the general purpose of these last figures will be sufficiently apparent without further remark.

In reference, however, to Figs. 21 and 22, where the arc of increasing curvature appears as described with the radius $a.S.$ it is desirable to point out more particularly the circumstance (already stated) that the curve is not an arc described with a uniform radius and a fixed centre. The curve may be considered as compounded of fractional arcs of which the centres are not co-incident, or as belonging to an ellipse. The same remark would apply to the arc of curvature, Fig. 25, if described from the same centre $a.$ with the increased radius $a.\mathfrak{X}.$ For example (Fig. 22)—If the distance of $a.$ from $B.$ in Fig. 22, be made two tenths of $B.D.$, and $B.\mathfrak{X} - Ba$ $(= 7{\cdot}854 - 2) = 5{\cdot}854$ be taken as the radius, and the arc be described from $\mathfrak{X}.$ as the original point, the arc will then fall inside $\mathfrak{X}.S.$, because the distance of $S.$ from $a.$ is $5{\cdot}856$, consequently

the radius will fall short by the difference of about 2 parts in six thousand. Similarly, in Fig. 25, (taking the point *a.* at the same distance, *B.a.* as before,) $B.\mathfrak{X}. - B.a. = 7·85674 - 2 = 5·85674$; the arc now falls beyond $\mathfrak{X}.S.$, because the distance *a.S.* is as before 5·8560, and the radius is now 5·85674.

But, if we bisect the line $S.\mathfrak{X}.$, (the extremity of the line $R.Y.\mathfrak{X}.$, Fig. 25,) and from the point of bisection at right angles to $S.\mathfrak{X}.$ we draw a line intercepting the line *B.D.*, the point of interception will be *b.* at a distance from *B.* = 2·00452.

Therefore $B.\mathfrak{X}. - B.b. = 7·85674 - 2·00452 = 5·85223$. . which is also the exact distance of the point *S.* at the extremity of the half quadrant *B.S.*, from the point *b.* on the line *B.D.* Wherefore, in this case, we find the arc $\mathfrak{X}.S.$ is the true arc of a circle described from the centre *b.* with the radius $b.\mathfrak{X}. = 5·85223$. . .

We have brought under consideration four arcs, namely:—

(1) The arc of the arc-length $\mathfrak{X}.\ \mathfrak{X}.$

(2) The complementary arc (or, arc of curvature) $S.\mathfrak{X}.$

(3) The arc of the ultimate sine-length $\mathfrak{X}.\ \mathfrak{X}.$

(4) The arc of increasing curvature $S.\ \mathfrak{X}.$

The first and third, indicated in the Figures by the same letter $\mathfrak{X}.\ \mathfrak{X}.$, because too close to each other to be clearly distinguishable, are both of them arcs of a circle described from the same centre *B.* The radius of the one (1) equalling 7·85674...&c.; and the radius of the other (3) equalling 7·85398...&c.

The second and fourth arcs we wish now again to compare and contrast, in order to distinguish clearly between them. The fourth is not the arc of a circle, but a curve of a compound character; as already mentioned, it may be considered as compounded of (ultimate) fractional curves differing (very slightly) from each other, or (preferably) as a compound arc described from a centre, the relative position of which is constantly varying, the posi-

tion and variation thereof being definite relatively to each ultimate fraction of the arc described.

The following table exhibits the comparison throughout the greater part of the curve:—

A. are the sine-lengths of the successive arcs—obtained by the duplication of the successive fractions of the repeatedly bisected arc—by the terminal extremities of which 'the compound arc of increasing curvature' is formed.*

B. are the sine-lengths of the equivalent successive arcs (corresponding to those of column *A.*)—resulting from the continual 'unbending' or 'straightening' of the primary arc—by the terminal extremities of which 'the complementary arc of absolute curvature' is formed.

	A.	*B.*
$\frac{C}{8}$	7·071068	7·071068
$\frac{C}{16}$	7·653668	7·65517
$\frac{C}{32}$	7·803613	7·80606
$\frac{C}{64}$	7·841371	7·84405
$\frac{C}{128}$	7·850828	7·85356
$\frac{C}{256}$	7·853193	7·85539
$\frac{C}{512}$	7·853785	7·85655
$\frac{C}{1024}$	7·853937	7·85670
$\frac{C}{2048}$	7·853969	7·85673
Finally	7·85398...&c.	7·85674...&c.

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* These sine lengths agree with those of the table at page 55 of Appendix, Part Second.

Herein we observe (1) a quite independent verification of the fact, already amply demonstrated, that 7·85674 is the actual arc-length of the half-quadrant *B.S.;* and moreover we observe (2) a very interesting definite relationship established between the arc $\mathscr{X}.S.$ (Fig. 25) and the half quadrant *B.S.;* namely the magnitudinal ratio of the circles to which they respectively belong is . . as $\mathscr{X}.S.$: *B.S.* : : radius $\mathscr{X}.b$, 5·85223 : radius *A.B.*, 10·0000, (i.e., a half quadrant described with the lesser radius would have that ratio to the arc *B.S.*,)

Note.—Referring to Fig. 25, the following will indicate the elements of the computation :—

Bisecting the line $S.\mathscr{X}.$ and drawing the line *d.b.* intercepting *B.D.* at *b.* we obtain $\mathscr{X}.d.$ as the base of a triangle similar to the triangle $S.\mathscr{X}.W.$; therefore

as $\mathscr{X}.W.$: $\mathscr{X}.S.$: : $\mathscr{X}.d.$: $\mathscr{X}.b.$

Now since $S.W. = 2{\cdot}92893$ $\quad S.\mathscr{X}. = 3{\cdot}03247$

And $\mathscr{X}.d. = S.\mathscr{X}. \div 2 = 1{\cdot}516235$, therefore

·785674 : 3·03247 : : 1·516235 : 5·85223..,

which last number is the length of the radius $b.\mathscr{X}.$

$$7{\cdot}85674 - 585222 = 200452 = B.b.$$

But $7{\cdot}07107 - 2{\cdot}00452 = 5{\cdot}06655$

And $\sqrt{5{\cdot}06655^2 + 2{\cdot}92893^2} = 5{\cdot}85223$..,

that is, the square of ‘the sine of the half-quadrant less *B.b.*,’ added to the square of the versed sine *S.W.* gives the square of the radius $b.\mathscr{X}.$

In like manner the sine-lengths of the successive equivalent arcs of decreasing curvature (column *B.*) may be verified.

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ILLUSTRATION TO

Fig. 20 (R.)

Analytical Examination of the quadrant and of the arc of increasing curvature.

1—Fig. 20 (R.) *The radius A.B. as equalling* 10.

The radius A.'a. : A.N. :: A.'d. : A.B. Consequently A.'a . A.'d :: A.N. : A.B. And the arcs, sines, tangents, and their equal divisions, and other related lines, belonging to each of these radii, are in the same ratios respectively to the similar lines belonging to each of the other radii; that is,... the arc 'a.b. : N.H. :: 'd.e. : B.S., the sine 'c.b. : d.H. :: 'a.e. : N.S., and so on. (Observe that the radius of the arc N.H. = the cosine of B.S.)

A.B. = 10·00 A.N. = 7·07107.
A.d. = 5·00 A.'a. = 3·535535.

As C.'a. : 'a.d. :: d.N. : N.B., and therefore, inversely, as 2·92893 : 2·07107 :: 1·464465 : 1·035535.

And as b.e. : e.H. :: H.S. : S.D. *i.e.*, (inversely,) as 4·14214 : 2·92893 :: 2·07107 : 1·464465.

Consequently 4·14214 : 2·92893 :: 2·92893 : 2·07107; &c., &c.

2—Fig. 20 (R.) *The radius A.B.* = 10.

A.e. = e.g. = B.V. = K.B. = K.S. = S.D. = 4·14214.
A T. = B.Q. = C.R. = D.Z. = 2 B.V. = 8·28428.
N.K. = K.H. = N.B. (the versd. sine of B.S.) = 2·92893.
Z.Y. = R.V. = &c., &c. = 4·14214.
B.R. = A.Z. = D.T. = C.Q. = B.e. = 10·00 − 4·14214.
= 5·85786 (= Cosine of half-quadrant described with A.g.)

Because the triangle A.N.K., is a part of the similar triangle A.B.V., N.K. : B.V. :: A.N. : A.B. Therefore:—

2·92893 : 4·14214 :: 7·07107 : 10·000.
2 92893 : 7·07107 :: 4·14214 : 10·000.
2·92893 × 10·000 = 4·14214 × 7·07107.

The quadrant e.y. (=g.Z.) : B.C. :: A e. : A.B.

The arc e.I. (=g.K.) : B.S. :: A.e. : A.B. :: 4·14214 : 10·00.

But the proportion e.y. (or g.z.) : B.S. :: 8·28428 : 10·000, is not strictly correct, because e.y., and g.Z., are quadrants, and contain twice the proportionate amount of curvature contained in the half-quadrant B.S. (See appendix to Part Third.)

Draw g.W. the tangent of the half-quadrant g.K. g.W. = B.V. the tangent of the arc B.U. of 22½ degrees.

Scholium.—Observe the relationship... e.g. : A.B. :: 4·14214 : 10. And the tangent to the half-quadrant, of which e.g. is the radius, equals the tangent to the arc of 22½ degrees of which A.B. is the radius; but the sine (N.K.) of g.K. : the sine of B.U. :: A.K. : A.U.

Some of the more immediate relations of these (quantitive magnitudes) numbers may be noted,—as for instance:—

$7{\cdot}07107 \times 2 = 14{\cdot}14214 \qquad 14{\cdot}14214 - 10 = 4{\cdot}14214.$

$7{\cdot}07107^2 = 50{\cdot}0000 \qquad 4{\cdot}14214 \times 2 = 8{\cdot}28428.$

$7{\cdot}07107 - 5{\cdot}000 = 2{\cdot}07107$ (= half the tangent of $\frac{C.}{16}$)

$14{\cdot}14214^2 = 200{\cdot}00 \qquad 1{\cdot}414214^2 = 2{\cdot}000.$

$8{\cdot}28428 \times 2{\cdot}07107 = 17{\cdot}15732.$

$4{\cdot}14214^2 = 17.15732.$

$8{\cdot}28428 \times 7{\cdot}07107 = 58{\cdot}5786.$

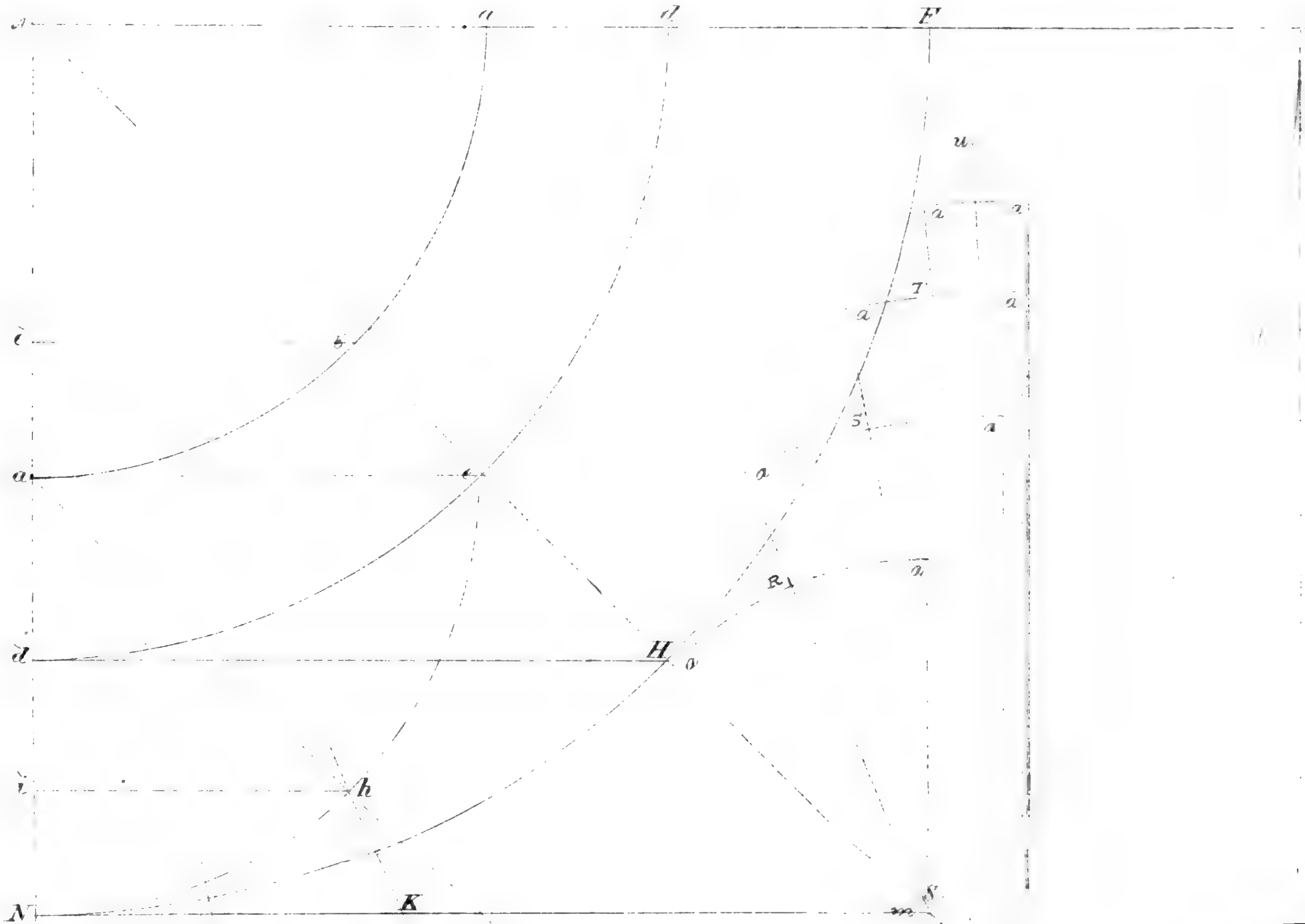
$2{\cdot}92893 \times 2 \times 10 = 58{\cdot}5786.$

$2{\cdot}92893^2 = 8{\cdot}5786.$

$7{\cdot}653667^2 = 58{\cdot}5786.$

And so on. (The last number represents the duplicated sine of $\frac{C.}{16}$)

1_FIG. 20/R.



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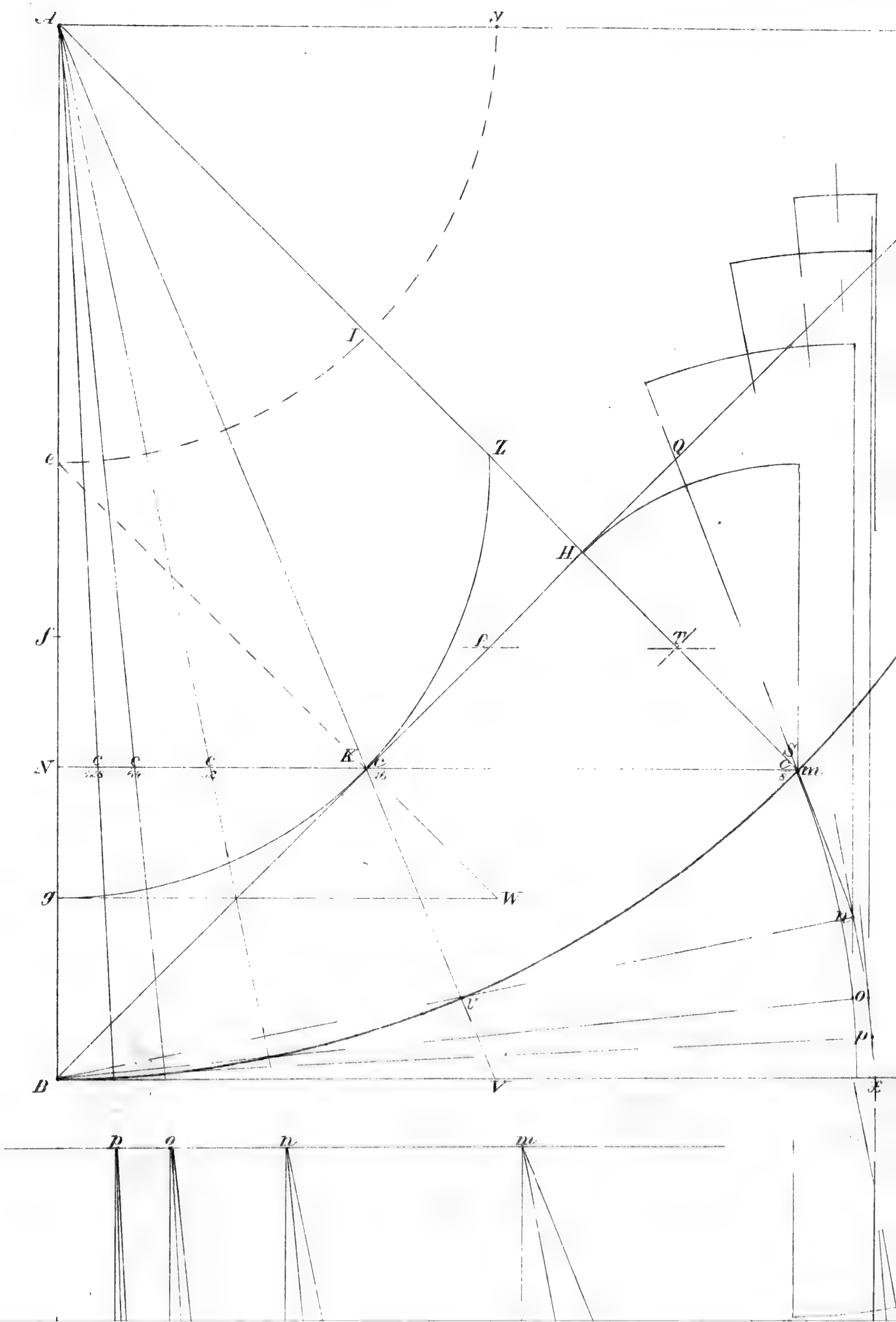
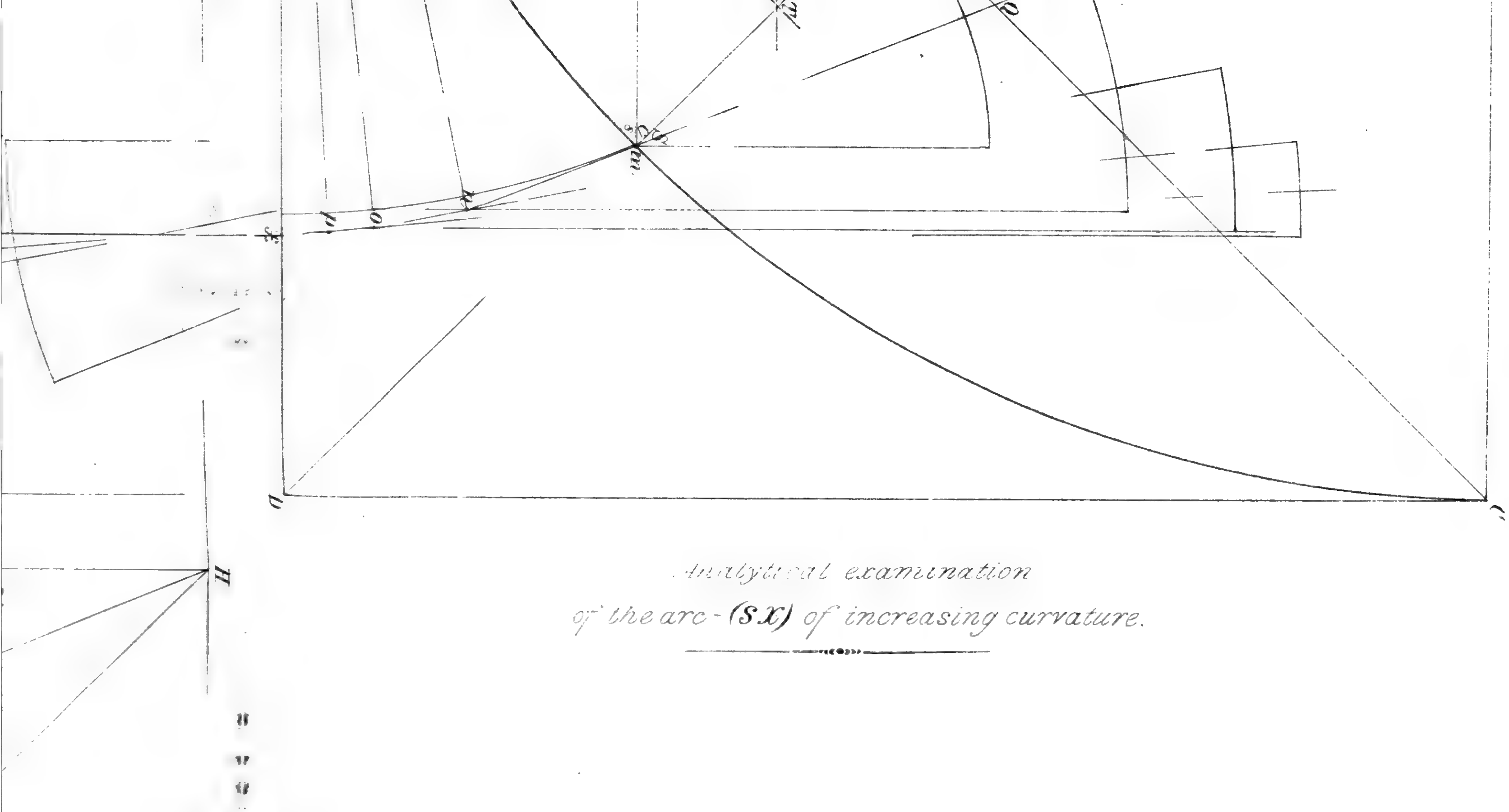
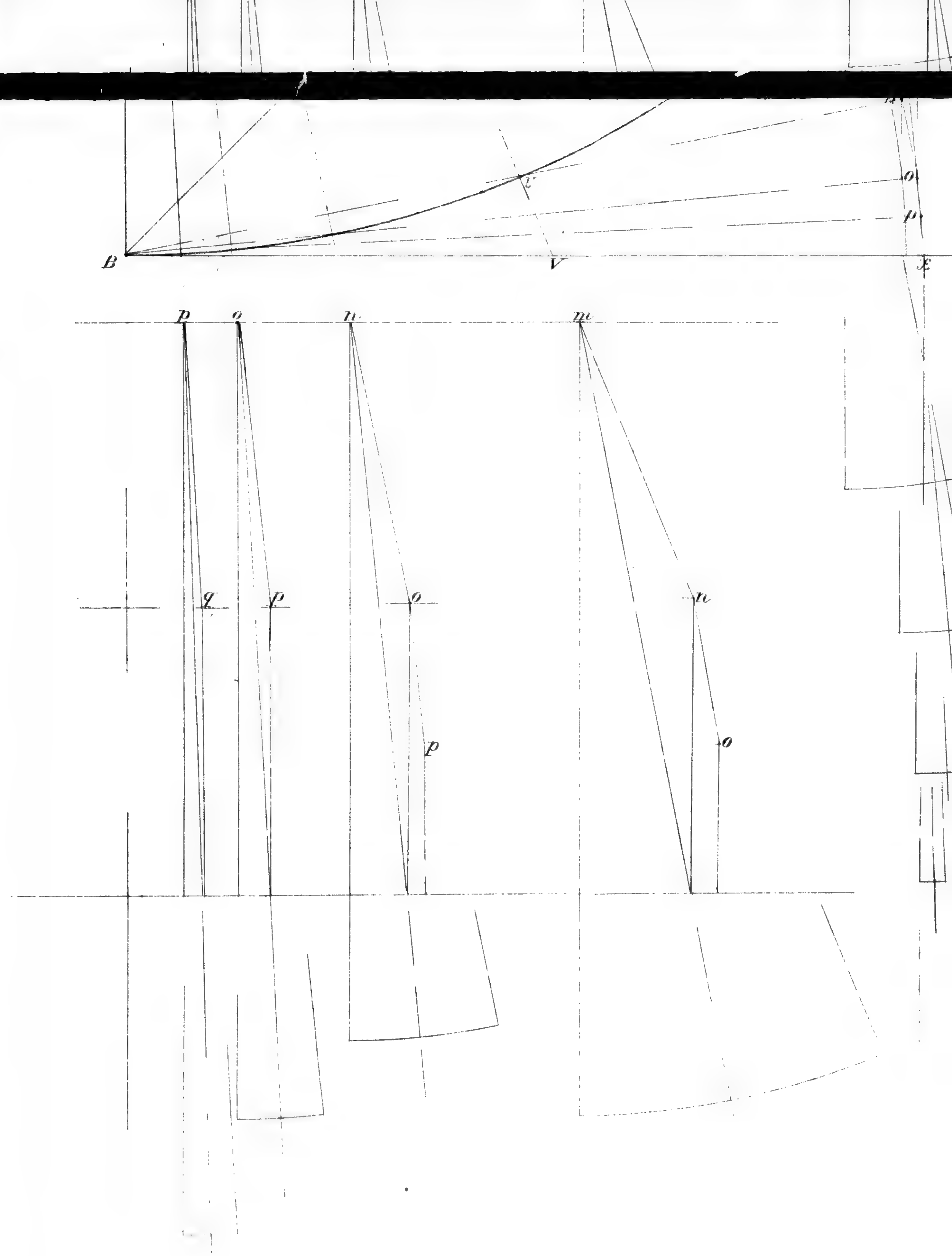
2. FIG. 20. (R)





Analytical examination
of the arc - (SX) of increasing curvature.



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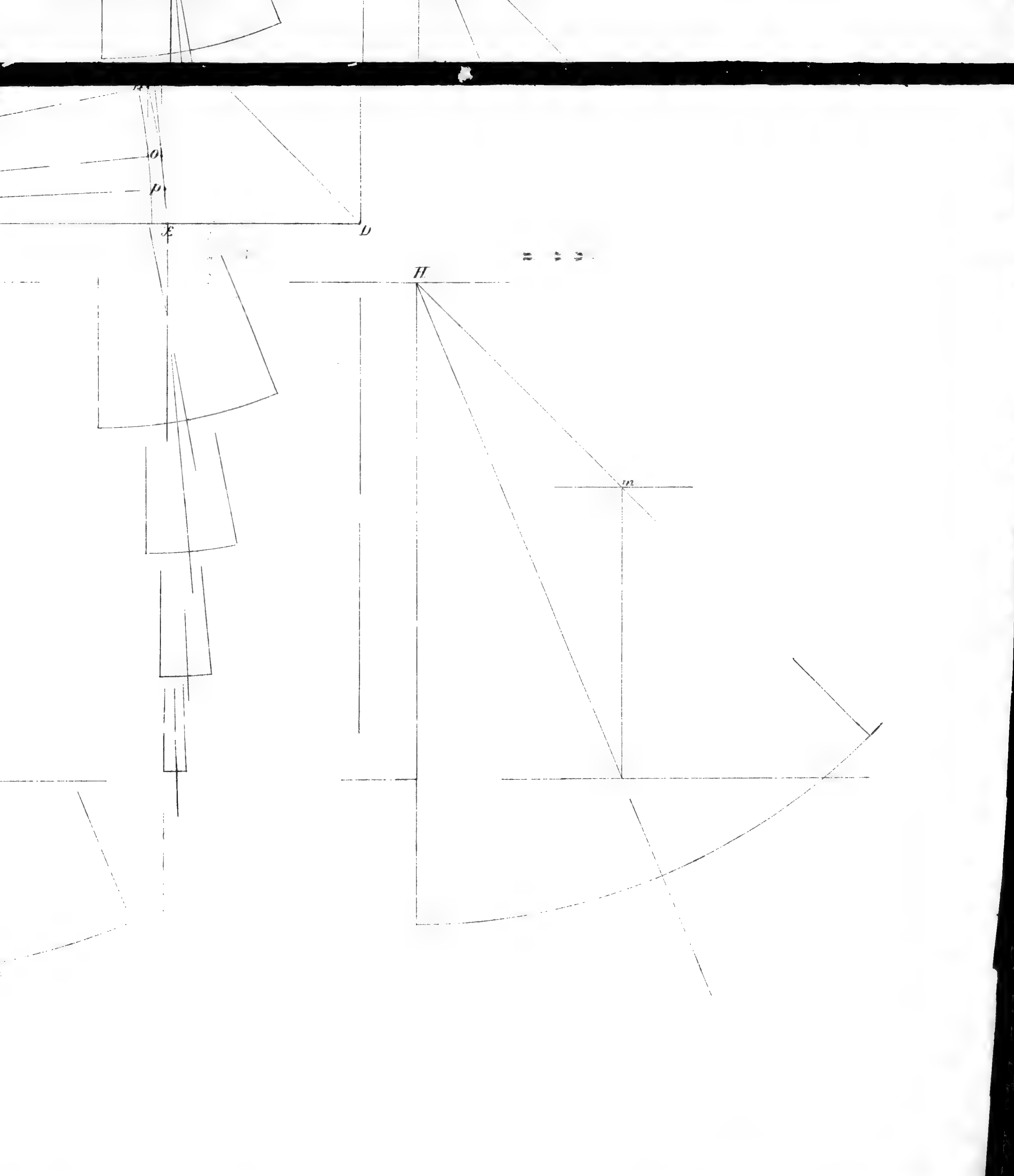
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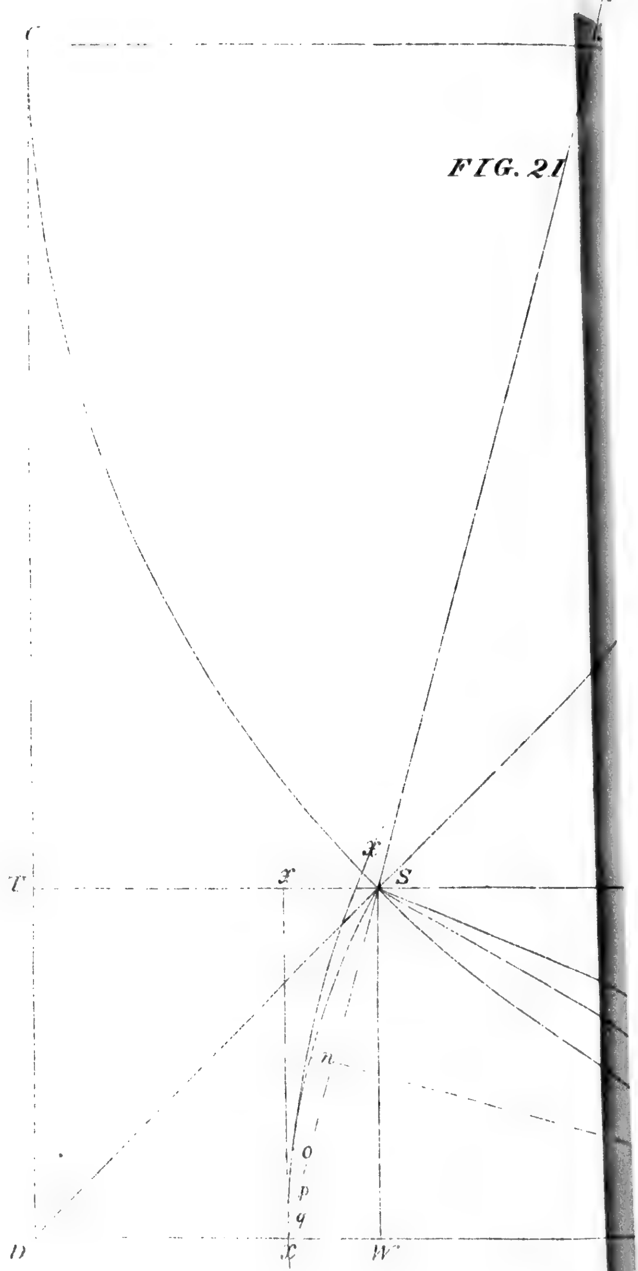
FIG. 21



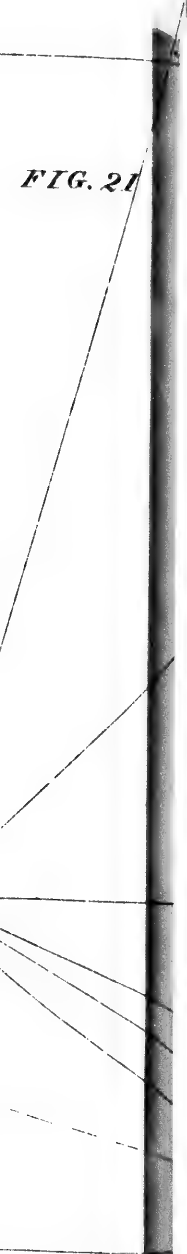
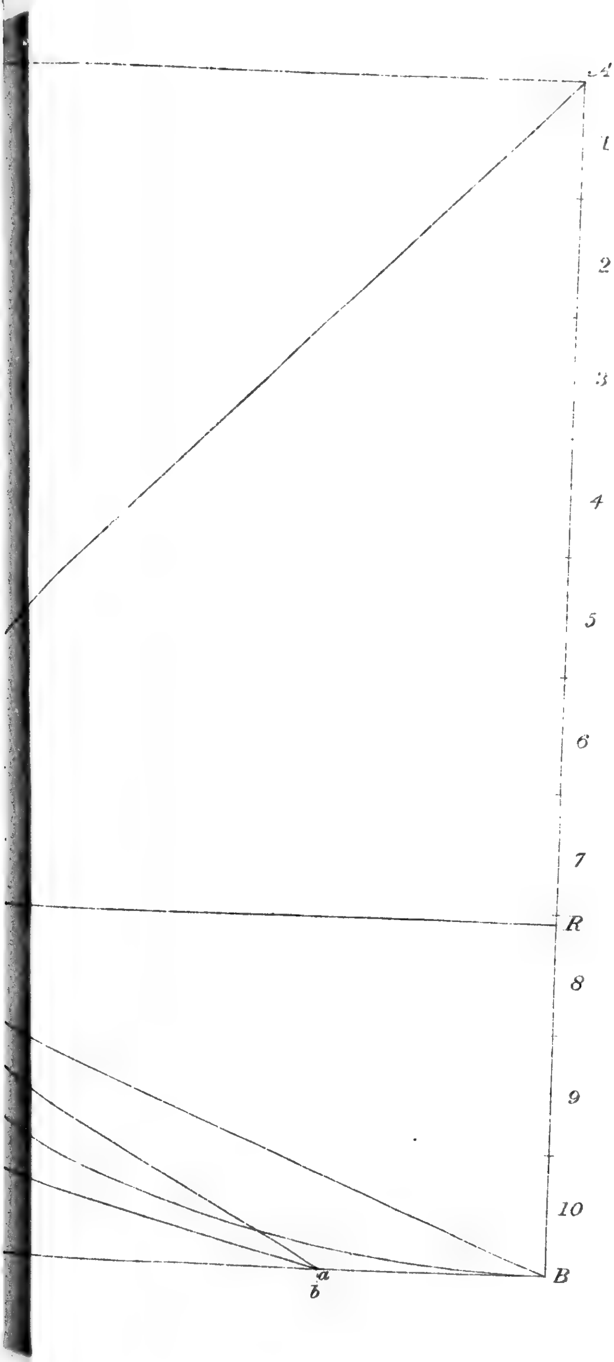


FIG. 21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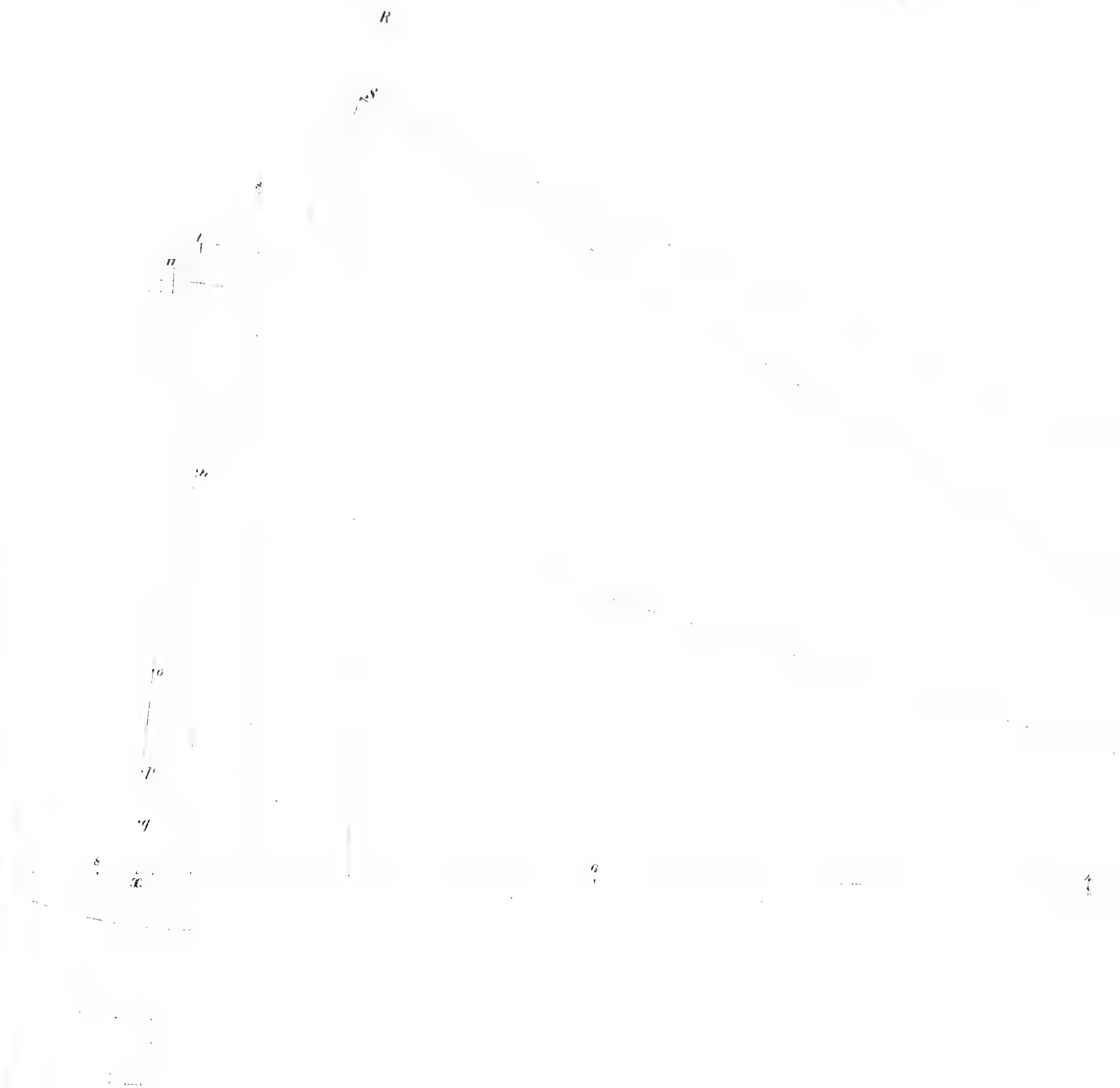
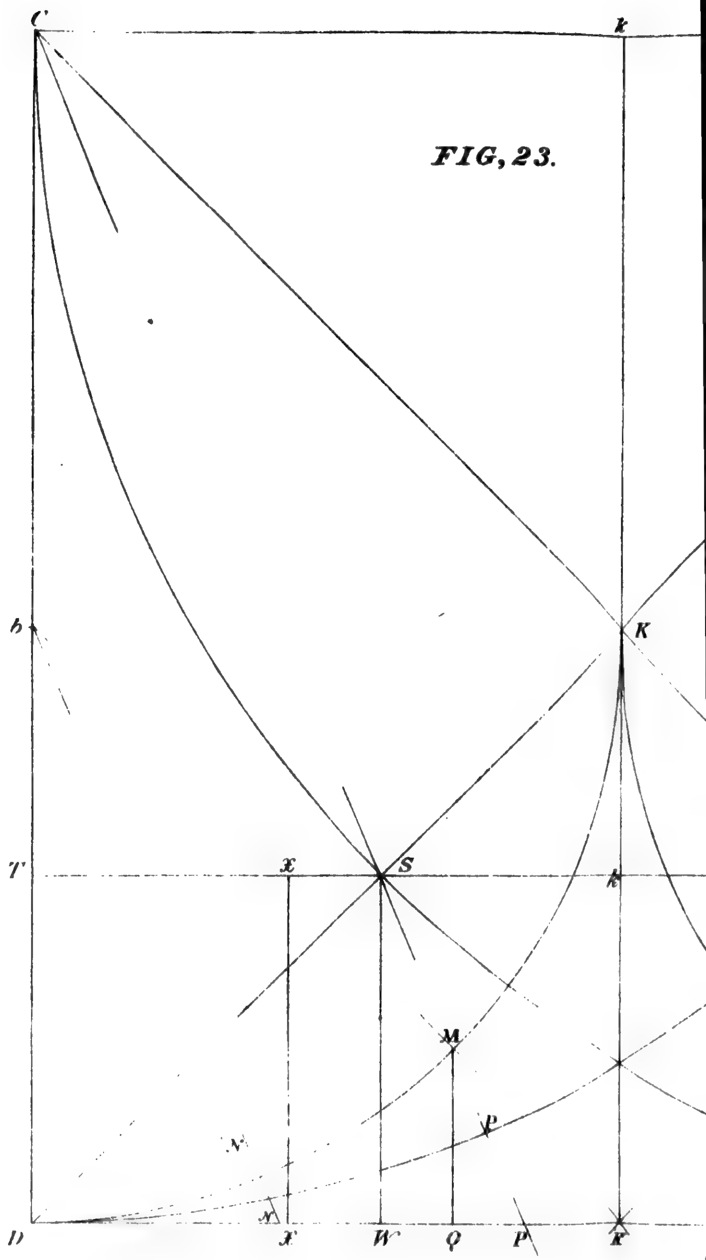


Fig. 22.

FIG, 23.



FIG, 23.

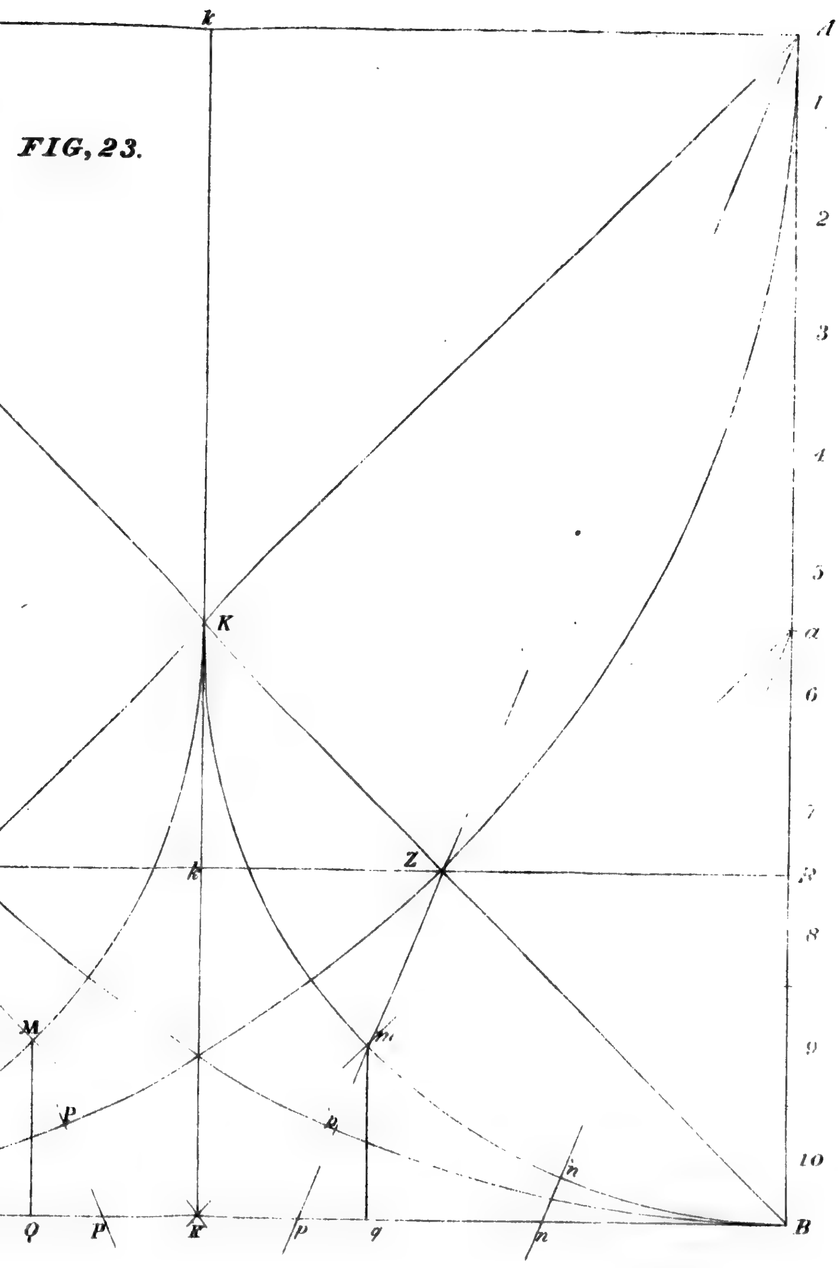


Plate 9

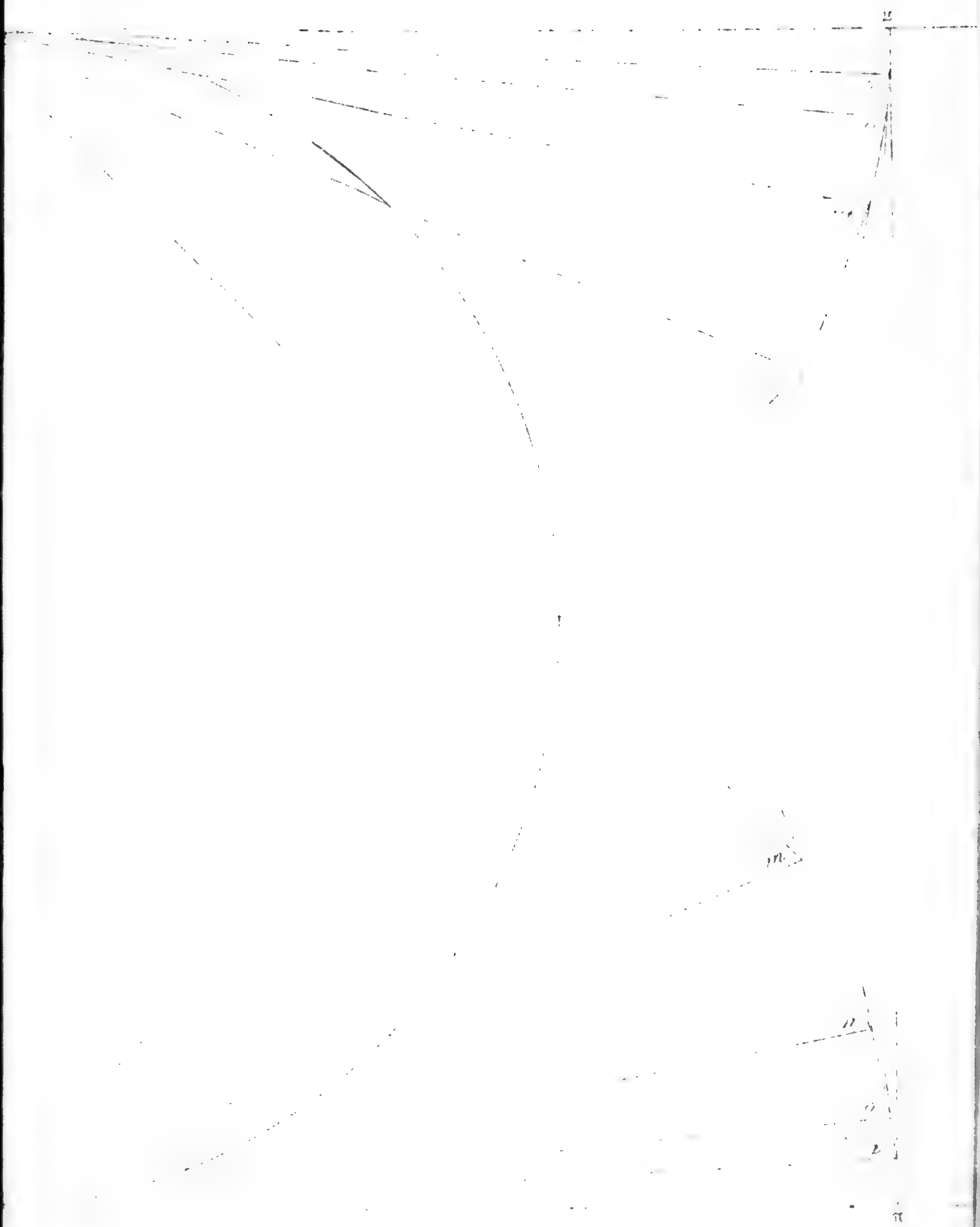


Fig 30.

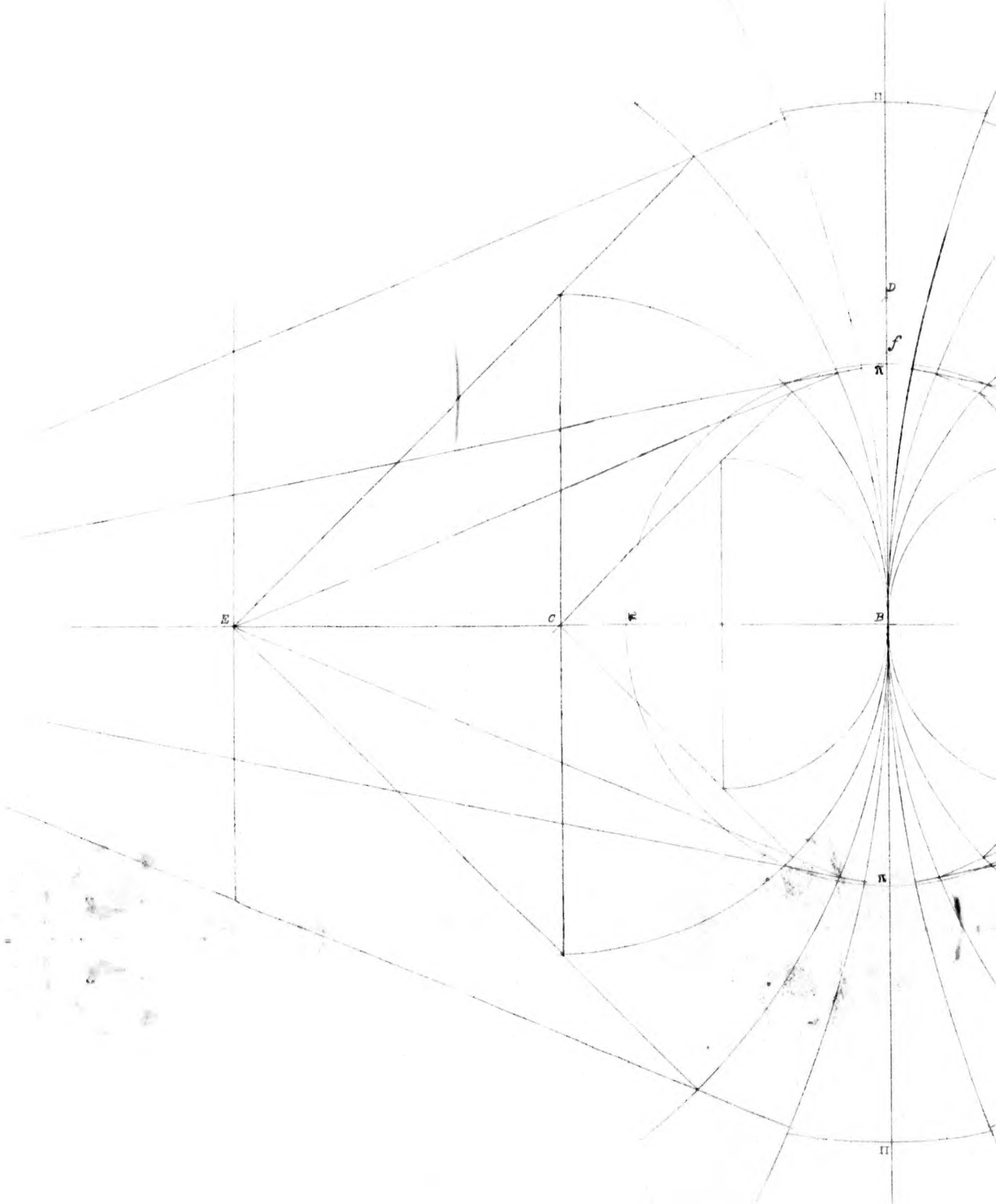


Fig. 30.